THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH2060B Mathematical Analysis II (Spring 2017)
Tutorial 6

Tongou Yang

1. (Riemann Sum) Let $f : [a, b] \to \mathbb{R}$ be bounded.
   (a) Define tagged partition and Riemann sums.
   (b) State the definition of Riemann integrability in terms of Riemann sums.
   (c) State the Cauchy criterion of Riemann integrability.
   (d) Using this definition, show that the Dirichlet function on $[0, 1]$ is not Riemann integrable.
   (e) State the Equivalence theorem.

2. (a) State two forms of the fundamental theorem of calculus.
   (b) Let $f \in \mathcal{R}[a, b]$. Show that if $f : [a, b] \to \mathbb{R}$ is continuous at $c \in (a, b)$, then
   $$\lim_{r\to 0^+} \frac{1}{2r} \int_{c-r}^{c+r} f(y)dy = f(c).$$
   Find an example such that the above holds, but f is not continuous at c.
   (c) Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and $G, H : \mathbb{R} \to \mathbb{R}$ be differentiable everywhere. Compute the derivative of
   $$\phi(x) := \int_{G(x)}^{H(x)} f(t)dt$$
   (d) Prove that there is no function $f : \mathbb{R} \to \mathbb{R}$ which is continuously differentiable everywhere and satisfies $f(0) = 0, f(1) = 2, |f'(x)| \le 1.999$ for any $x \in \mathbb{R}$. If we use Mean Value theorem, then the "continuously" can be removed.

3. (Vanishing Lemmas)
   (a) Let $f : [a, b] \to \mathbb{R}$ be continuous and satisfy $\int_a^x f(t)dt = 0$ for any $x \in [a, b]$. Show that f is identically zero.
   (b) Let $f : [a, b] \to \mathbb{R}$ be continuous and $f \ge 0$. Suppose $\int_a^b f(x)dx = 0$. Show that f is identically zero.

4. Not Enough Time to Talk About
   (a) State the change of variable formula.
   (b) Let φ be strictly positive and continuously differentiable on $\mathbb{R}$, and $x > 0$. Using change of variable formula, show that
   $$\int_0^x \frac{\varphi'(t)}{\varphi(t)}dt = \ln \varphi(x) - \ln f(0)$$
   (Q: Why is left hand side Riemann integrable?)

(c) State the integration by parts formula.

(d) Using integration by parts, show that the following limit of integral converges:

$$\lim_{R\to\infty}\int_0^R \frac{\sin x}{x}dx$$

Remember that we should first show that for each $R > 0$, the function $\frac{\sin x}{x}$ is Riemann integrable on $[0, R]$.

5. (Symmetry) Let all functions be defined on $\mathbb{R}$ and smooth, and $-\infty < a < b < \infty$.

(a) Show that the derivative of an odd function is even and that the derivative of an even function is odd.

(b) Consider the indefinite integral:

$$F(x) := \int_0^x f(t)dt$$

If f is even, then F is odd; if f is odd, then F is even.

(c) (Translation Invariant) Let $t \in \mathbb{R}$.

$$\int_a^b f(x)dx = \int_{a+t}^{b+t} f(x-t)dx.$$

(d) Let f be any periodic function of period $T > 0$, that is, for any $x \in \mathbb{R}$, we have:

$$f(x+T) = f(x)$$

Then

$$\int_a^{a+T} f(x)dx = \int_0^T f(x)dx,$$

for any $a \in \mathbb{R}$.

Proof. Given a, there are unique $a' \in [0, T)$, $k \in \mathbb{Z}$ with $a - a' = kT$. By periodicity,

$$\int_a^{a+T} f(x)dx = \int_a^{a+T} f(x-kT)dx = \int_{a'}^{a'+T} f(y)dy, \text{ by change of variable}$$

Hence it suffices to assume that $a \in [0, T)$. We split the integral into

$$\int_a^{a+T} f(x)dx = \int_a^T f(x)dx + \int_T^{a+T} f(x)dx$$

Note that by periodicity again,

$$\int_T^{a+T} f(x)dx = \int_T^{a+T} f(x-T)dx = \int_0^a f(y)dy, \text{ by change of variable}$$

Hence

$$\int_a^{a+T} f(x)dx = \int_a^T f(x)dx + \int_0^a f(y)dy = \int_0^T f(x)dx.$$

$\square$